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On clique separators, nearly chordal graphs, and the Maximum Weight Stable Set Problem

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Abstract

Clique separators in graphs are a helpful tool used by Tarjan as a divide-and-conquer approach for solving various graph problems such as the Maximum Weight Stable Set (MWS) Problem, Maximum Clique, Graph Coloring and Minimum Fill-in, but few examples of graph classes having clique separators are known. We use this method to solve MWS in polynomial time for two classes where the unweighted Maximum Stable Set (MS) Problem is solvable in polynomial time by augmenting techniques but the complexity of the MWS problem was open. Another example, namely a result by Alekseev for the MWS problem on a subclass of P_5-free graphs obtained by clique separators, can be improved by our techniques. We also combine clique separators with decomposition by homogeneous sets in graphs and use the following notion: A graph is *nearly Π* if for each of its vertices, the subgraph induced by the set of its nonneighbors has property Π. We deal with the cases $\Pi \in \{\text{chordal}, \text{perfect}\}$. This also simplifies a result obtained by a method called struction.

Keywords: Nearly chordal graphs; Clique separators; Graph algorithms; Struction; Maximum Weight Stable Set Problem

1. Introduction

For a graph $G = (V, E)$ and a vertex weight function w on V, let $\alpha_w(G)$ ($\alpha(G)$) denote the maximum weight (maximum cardinality) of a stable vertex set in G.

The *Maximum Weight Stable* (or *Independent*) *Set* (*MWS*) Problem asks for a stable set of maximum weight in the given graph G with vertex weight function w. The *MS problem* is the MWS problem if all vertices v have the same weight $w(v) = 1$.

The M(W)S problem is one of the fundamental algorithmic graph problems known to be NP-complete in general and solvable in polynomial time on various graph classes by various techniques. Some of these techniques such as

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augmenting, reducing α-redundant vertices and a method called *struction* introduced by Ebenegger, Hammer, and de Werra in [14] lead only to efficient solutions of the unweighted Maximum Stable Set (MS) Problem.

A famous divide-and-conquer approach by using clique separators is described by Tarjan in [33]. It works for various problems on graphs such as Minimum fill-in, Coloring, Maximum Clique, and the MWS problem as shown in [33]. The subgraphs not containing clique separators are called *atoms* in [33]. Whenever MWS is efficiently solvable on the atoms of a graph G, it is efficiently solvable on G. However, few examples are known where this approach could be applied for obtaining a polynomial time MWS algorithm on a graph class. One of the examples is given by Alekseev in [2] showing that for (P_5, Q)-free graphs, the atoms are $3K_2$-free from which it follows that the MWS problem is solvable in polynomial time for (P_5, Q)-free graphs. We will improve this result and explain it in terms of perfect graphs.

We also use clique separators in combination with nearly chordal graphs for the MWS problem on two classes where the complexity of the MWS problem remained open:

(i) for (P_6, C_4)-free graphs which considerably extends and improves the polynomial time result for the MS problem given in [28] by using augmenting techniques;
(ii) for (P_5, P)-free graphs which extends and improves a result in [11,25] where a (robust, see [11]) polynomial time solution for the MS problem on (P_5, P)-free graphs was given using α-redundant vertices in [11] and an augmenting argument in [25], respectively.

The class of (P_6, C_4)-free graphs contains split graphs and $(C_4, 2K_2)$-free graphs, and (P_5, P)-free graphs also generalize various interesting graph classes such as split graphs, $(C_4, 2K_2)$-free graphs, co-bipartite graphs, cographs and P_4-sparse graphs (for the definition of all these classes see e.g. [10]). In Theorem 7, we show that for (P_6, C_4)-free graphs without clique separator, the MWS problem is efficiently solvable. This leads to a polynomial time MWS algorithm on this class by Tarjan's approach.

We also combine clique separators with homogeneous sets in order to refine graph decomposition, and we use the following notion: Let Π denote a graph property. A graph is *nearly* Π if for each of its vertices, the subgraph induced by the set of its nonneighbors has property Π. We deal with the cases $\Pi \in$ {chordal, perfect}.

Obviously, the MWS problem on a graph G with vertex weight function w can be reduced to the same problem on antineighborhoods of vertices in the following way:

$$\alpha_w(G) = \max\{w(v) + \alpha_w(G[\overline{N}(v)]) \mid v \in V\}.$$

Thus, whenever MWS is solvable in time T on a class with property Π then it is solvable on nearly Π graphs in time nT. For example, Frank [17] gave a linear time algorithm for the MWS problem on chordal graphs. Hence, the MWS problem can be solved in time $\mathcal{O}(nm)$ for nearly chordal graphs. Grötschel, Lovász and Schrijver [19] gave a polynomial time algorithm for the MWS problem on perfect graphs. Thus, the MWS problem can be solved in polynomial time for nearly perfect graphs.

Most of the results of our paper have been published as an extended abstract without proofs [8] in the Proceedings of IPCO 2005. The extended abstract [9] states the results in the section on (P_5, Q)-free graphs without proofs. We present the proofs in Section 5.

2. Basic notions

Throughout this note, let $G = (V, E)$ be a finite undirected graph without self-loops and multiple edges and let $|V| = n$, $|E| = m$. Let $V(G) = V$ denote the vertex set of graph G. For a vertex $v \in V$, let $N(v) = \{u \mid uv \in E\}$ denote the (*open*) *neighborhood* of v in G, let $N[v] = \{v\} \cup \{u \mid uv \in E\}$ denote the (*closed*) *neighborhood* of v in G, and for a subset $U \subseteq V$ and a vertex $v \notin U$, let $N_U(v) = \{u \mid u \in U, uv \in E\}$ denote the *neighborhood* of v with respect to U. The *antineighborhood* or *nonneighborhood* $\overline{N}(v)$ of a vertex v is the set $V \setminus N[v]$ of vertices different from v which are nonadjacent to v.

Disjoint vertex sets X, Y *form a join*, denoted by $X \textcircled{1} Y$ (*co-join*, denoted by $X \textcircled{0} Y$) if for all pairs $x \in X$, $y \in Y$, $xy \in E$ ($xy \notin E$) holds. We will also say that X has a join to Y, that there is a join between X and Y, or that X and Y are connected by join (and similarly for co-join). Subsequently, we will consider join and co-join also as operations, i.e., the co-join operation for disjoint vertex sets X and Y is the disjoint union of the subgraphs induced by X and

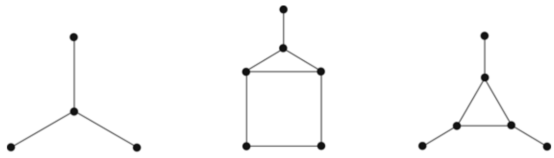

Fig. 1. The claw, antena and net.

Y (without edges between them), and the join operation for X and Y consists of the co-join operation for X and Y followed by adding all edges $xy \in E$, $x \in X$, $y \in Y$.

A vertex $z \in V$ *distinguishes* vertices $x, y \in V$ if $zx \in E$ and $zy \notin E$ or $zx \notin E$ and $zy \in E$. We also say that *a vertex z distinguishes a vertex set $U \subseteq V$*, $z \notin U$, if z has a neighbor and a nonneighbor in U. A vertex set $M \subseteq V$ is a *module* if no vertex from $V \setminus M$ distinguishes two vertices from M, i.e., every vertex $v \in V \setminus M$ has either a join or a co-join to M.

A graph G is *prime* if it contains only trivial modules, i.e., $\emptyset$, $V(G)$ and one-elementary vertex sets. A nontrivial module is called a *homogeneous set*. The notion of module plays a crucial role in the *modular* (or *substitution*) *decomposition* of graphs (and other discrete structures) which is of basic importance for the design of efficient algorithms — see e.g. [27] for modular decomposition of discrete structures and its algorithmic use and [26] for a linear-time algorithm constructing the modular decomposition tree of a given graph.

For $U \subseteq V$, let $G[U]$ denote the subgraph of G induced by U. Throughout this paper, all subgraphs are understood to be induced subgraphs. Let $\mathcal{F}$ denote a set of graphs. A graph G is *$\mathcal{F}$-free* if none of its induced subgraphs is in $\mathcal{F}$.

A vertex set $U \subseteq V$ is *stable* (or *independent*) in G if the vertices in U are pairwise nonadjacent. For a given graph with vertex weights, the Maximum Weight Stable Set (MWS) Problem asks for a stable set of maximum vertex weight.

Let co-$G = \overline{G} = (V, \overline{E})$ denote the *complement graph* of G. A vertex set $U \subseteq V$ is a *clique* in G if U is a stable set in $\overline{G}$. Let K_ℓ denote the clique with ℓ vertices, and let ℓK_1 denote the stable set with ℓ vertices. K_3 is called *triangle*.

A *clique separator* (or *clique cutset*) in G is a clique K such that $G[V \setminus K]$ has more connected components than G.

For $k \geq 1$, let P_k denote a chordless path with k vertices and $k-1$ edges, and for $k \geq 3$, let C_k denote a chordless cycle with k vertices and k edges. A *hole* is a C_k with $k \geq 5$, and an *antihole* is $\overline{C_k}$ with $k \geq 5$. An *odd hole* (*odd antihole*, respectively) is a hole (antihole, respectively) with odd number of vertices.

For a subgraph H of G, a vertex not in H is a *k-vertex for H* if it has exactly k neighbors in H. We also say that H *has no k-vertex* if there is no k-vertex for H. For a set $S \subseteq V(H)$ with $|S| = k$ let M_S be the set of k-vertices for H adjacent to vertices in S. We also write $M_{a,b}$ respectively M_x for $S = \{a, b\}$ respectively $S = \{x\}$, etc. The subgraph H *dominates* the graph G if there is no 0-vertex for H in G.

A graph is *chordal* if it contains no induced C_k, $k \geq 4$. A graph is *nearly chordal* if for each of its vertices, the subgraph induced by the set of its nonneighbors is a chordal graph.

More generally, if Π is a graph property then a graph is *nearly Π* if for each of its vertices, the subgraph induced by the set of its nonneighbors has the property Π. Note that this notion appears in the literature in many variants, e.g., as nearly bipartite graphs [5].

3. Connected (claw, antena, net)-free graphs are nearly chordal

A graph is (*claw, antena, net*)*-free* (*CAN-free*) if it contains none of the graphs in Fig. 1 as induced subgraphs.

Theorem 1. *Connected* (*claw, antena, net*)*-free graphs are nearly chordal.*

Proof. Assume that there is a vertex $v \in V$ such that $G_v := G[V \setminus N[v]]$ is not chordal.

Case 1. G_v contains C_4. Let C be a C_4 in G_v with vertices v_1, v_2, v_3, v_4 and edges $v_i v_{i+1}$, $i \in \{1, 2, 3, 4\}$ (index arithmetic modulo 4). Since G is claw-free, C has no 1-vertex, and for any 2-vertex x of C, its neighbors in C are consecutive. Since G is antena-free, no 0-vertex is adjacent to a 2-vertex of C, and since G is claw-free, no 0-vertex is adjacent to a 3- or 4-vertex of C. Thus, C has no 0-vertex since G is connected but v is a 0-vertex for C — contradiction.

Case 2. G_v is C_4-free but contains C_k for $k \geq 5$. Let C be a C_k in G_v with vertices $v_1, \ldots, v_k$ and edges $v_i v_{i+1}$, $i \in \{1, \ldots, k\}$ (index arithmetic modulo k). As before, C has no 1-vertex, and 2- and 3-vertices of C have consecutive neighbors in C. 0-vertices are not adjacent to 2-vertices since G is net-free, and 0-vertices are not adjacent to k-vertices, $k \geq 3$ since G is claw-free. Thus, C has no 0-vertex but v is a 0-vertex of C — contradiction.

This shows that G is nearly chordal. □

Corollary 1. *The MWS problem can be solved in time $\mathcal{O}(nm)$ on (claw, antena, net)-free graphs.*

This simplifies and extends a result by Hammer, Mahadev and de Werra in [20] solving the MS problem in polynomial time by so-called *struction* (a quite complicated stability reduction method introduced by Ebenegger, Hammer, and de Werra in [14]) — CAN-free graphs have been one of the key examples for the use of struction. Another key example for the struction method was the larger class of (claw, net)-free graphs for which struction was shown to solve the MS problem in polynomial time in [21]; in [7], also this case was generalized in a similar way to a $\mathcal{O}(nm)$ algorithm for the MWS problem on the larger class of nearly (claw, AT)-free graphs. In the next section, we will show that (claw, net)-free graphs are nearly perfect. A third example is the class of circular-arc graphs for which struction solves the MS problem [18]; note, however, that circular-arc graphs are nearly interval graphs and thus, MWS can be solved in time $\mathcal{O}(nm)$ on circular-arc graphs (note that interval graphs are chordal).

It seems to be a challenging task to find examples where struction works well and cannot be replaced by such a simple technique.

4. Connected (claw, net)-free graphs are nearly perfect

Theorem 2. *Connected (claw, net)-free graphs are nearly hole-free.*

Proof. Let G be a connected (claw, net)-free graph. Suppose there is a vertex x such that in the nonneighborhood of x there is a hole C with vertices $v_1, v_2, \ldots, v_k$ and edges $v_i v_{i+1}$ for $i = 1, \ldots, k$ and $k \geq 5$ (with subscript taken modulo k). Since G is connected, we may assume x has a neighbor y that has neighbors in C. If y has more than two neighbors in C, then y is center of a claw. If y has exactly one neighbor v_i in C, then v_i is center of a claw. So, y has exactly two consecutive neighbors in C, say v_i and v_{i+1}. But now the vertices $v_{i-1}, v_i, v_{i+1}, v_{i+2}, x, y$ form a net. □

Theorem 3. *Connected (claw, net)-free graphs are nearly odd-antihole-free.*

Proof. Let G be a connected (claw, net)-free graph. Suppose there is a vertex x such that in the nonneighborhood of x there is an odd antihole C with vertices $v_1, v_2, \ldots, v_k$ and nonedges $v_i v_{i+1}$ for $i = 1, \ldots, k$ and k being an odd integer at least seven (with subscript taken modulo k.) Since G is connected, we may assume x has a neighbor y that has neighbors in C.

First, let us remark that y cannot be adjacent to two consecutive vertices v_i, v_{i+1} of C, for otherwise y is center of a claw. Now, let v_i be a neighbor of y in C. The above remark shows that y is not adjacent to v_{i-1}, v_{i+1}. If y is not adjacent to v_{i+2}, then y must be adjacent to v_{i+3} (for otherwise, v_i, v_{i+2}, v_{i+3}, y form a claw); but now the vertices $x, y, v_i, v_{i+1}, v_{i+2}, v_{i+3}$ form a net. We have shown that if y is adjacent to v_i, then y is adjacent to v_{i+2} and nonadjacent to v_{i+1}. It follows that k is even, a contradiction. □

Corollary 2. *Connected (claw, net)-free graphs are nearly perfect.*

Proof. By Theorem 2, prime (claw, net)-free graphs are nearly hole-free, and by Theorem 3, these graphs are nearly odd-antihole-free. Thus, by the Strong Perfect Graph Theorem by Chudnovsky, Robertson, Seymour and Thomas [12], these graphs are nearly perfect. □

Corollary 3. *The Maximum Weight Stable Set Problem can be solved in polynomial time for (claw, net)-free graphs.*

5. Atoms of (P_5, Q)-free graphs are nearly $(P_5, \overline{P_5}, C_5)$-free or 1-specific

Let Q denote the corresponding graph in Fig. 2. An *atom* of a graph G is an induced subgraph of G that contains no clique cutset. For an integer l, lK_2 denotes the union of l vertex-disjoint edges (the definition of 1-specific will be given later in this section.) In Theorem 4, Alekseev applied the clique separator technique to (P_5, Q)-free graphs:

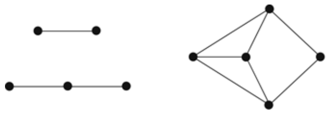

Fig. 2. The graph $P_2 + P_3$ and its complement called Q.

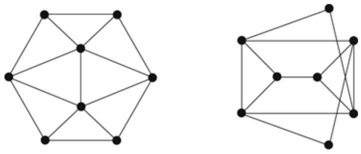

Fig. 3. A two-vertex extension xC_6 of the C_6 and its complement graph, the co-xC_6.

Theorem 4 (*Alekseev [2]*). *Atoms of* (P_5, Q)*-free graphs are* $3K_2$*-free.*

Farber in [15] has shown that a $2K_2$-free graph $G = (V, E)$ contains at most n^2 inclusion-maximal independent sets, $n = |V|$. Thus, the MWS problem on these graphs can be solved in time $\mathcal{O}(n^4)$ since Paull and Unger [29] gave a procedure that generates all maximal independent sets in a graph in $\mathcal{O}(n^2)$ time per generated set (see also [34,24]).

Farber's result has been generalized to $l \geq 2$: lK_2-free graphs have at most n^{2l-2} inclusion-maximal independent sets [1,4,16,30], and thus, MWS is solvable on lK_2-free graphs in time $\mathcal{O}(n^{2l})$.

Since $3K_2$-free graphs have at most n^4 maximal stable sets, the MWS problem is solvable in time $\mathcal{O}(n^8)$ on (P_5,Q)-free graphs by the clique cutset approach of Tarjan and a corresponding enumeration algorithm for all maximal stable sets in a $3K_2$-free graph. Theorem 4, however, does not give much structural insight. Our main result of this section, namely Theorem 5, shows the close connection of (P_5, Q)-free graphs to known classes of perfect graphs and in particular leads to a faster MWS algorithm. Preparing for this, we have to define a simple type of graphs which results from a certain extension of the $\overline{C_6}$ by two vertices (which we call co-xC_6) and the complement of this graph (xC_6, see Fig. 3).

A graph G is 1-*specific* if it consists of a co-xC_6 H, a stable set S with $S \subseteq G - H$ that consists of 2-vertices of H having the same neighbors as one of the degree 2 vertices in H, and a clique U of universal (i.e., adjacent to all other) vertices. Note that the MWS problem for 1-specific graphs can be solved in the obvious way.

Theorem 5. *Atoms of* (P_5, Q)*-free graphs are either nearly* $(P_5, \overline{P_5}, C_5)$*-free or* 1*-specific graphs.*

The proof of Theorem 5 is based on the subsequent Lemmas 1–3.

Lemma 1. *Atoms of* (P_5, Q)*-free graphs are nearly* $\overline{P_5}$*-free.*

Proof. Assume that there is a vertex $v \in V$ such that $G[\overline{N(v)}]$ contains a $\overline{P_5}$ H, say with vertices $v_1, \ldots, v_5$ such that $v_1, \ldots, v_4$ induce C_4 with edges $v_i v_{i+1}$ (index arithmetic modulo 4) and v_5 is adjacent to v_2 and v_3. Recall that a vertex not in H is a k-vertex of H, if it has exactly k neighbors in H.

Since G is (P_5, Q)-free, a straightforward case analysis shows that in $N(v)$, the C_4 in H has no 1-vertex and no 3-vertex, and the neighbors of 2-vertices are nonconsecutive in the C_4. This immediately implies that H has no 1- and no 2-vertex in $N(v)$.

Now, if x is a 3-vertex for H then x is adjacent to v_5 and thus x is a 2-vertex for the C_4 in H, say, $xv_1 \in E$ and $xv_3 \in E$, while $xv_2 \notin E$ and $xv_4 \notin E$, but now, $xv_1v_2v_3v_5$ induce a Q — a contradiction. Thus, H has no 3-vertex in $N(v)$.

A 4-vertex for H in $N(v)$ must be adjacent to $v_1, \ldots, v_4$ and not to v_5. Let F denote the set of 4-vertices for H in $N(v)$, let U denote the set of 5-vertices for H in G, and let $U_1 := U \cap N(v)$ and $U_0 := \{u \in U \setminus U_1 \mid u$ is adjacent to a 0-vertex for H in $N(v)\}$; obviously, F and U are cliques, and $F \textcircled{1} U_1$ since G is Q-free. Moreover, $F \textcircled{1} U_0$: Assume that $x \in F$, $u \in U_0$ with $xu \notin E$, and $uy \in E$ for a 0-vertex $y \in N(v)$. Then, since $xvyuv_5$ is no P_5, $xy \in E$ follows but now v_1v_2xyu is a Q — contradiction.

This implies that $F \cup U_1 \cup U_0$ is a clique. We claim that it separates v and H: Assume not; then there is a (shortest) path from v to H in $G[V \setminus (F \cup U_1 \cup U_0)]$ via a 0-vertex x having a neighbor $y \in V \setminus N(v)$ which is then adjacent to

H since G is P_5-free. Now it is clear that y cannot distinguish H since G is P_5-free. Thus, $y \in U_0$ which shows the claim. But now $F \cup U_1 \cup U_0$ is a clique separator — a contradiction which shows Lemma 1. □

Lemma 2. *Atoms of (P_5, Q)-free graphs containing an induced co-xC_6 are 1-specific graphs.*

Proof. Let G be a (P_5, Q)-free graph without clique cutset containing an induced co-xC_6 C, say with vertices $\{v_1, \dots, v_8\}$ such that $\{v_1, \dots, v_6\}$ induce $\overline{C_6}$ and v_7, v_8 are the vertices of degree 2 in the co-xC_6.

By Lemma 1, C has no 0- and no 1-vertex since every house dominates the graph. A long but straightforward case analysis shows that C has also no k-vertex for $k \in \{3, 4, 5, 6, 7\}$ and the 2-vertices have the same neighbors in C either as v_7 or as v_8. Let N_7 (N_8, respectively) denote the 2-vertices having the same neighbors as v_7 (v_8) in C. N_7 and N_8 are stable sets and there is a co-join between them: $N_7 ⓪ N_8$.

Let U denote the set of 8-vertices of C. Since G is Q-free, U is a clique and there is a join between U and the sets of 2-vertices: $U ① (N_7 \cup N_8)$. Thus, G is 1-specific which shows Lemma 2. □

Lemma 3. *Atoms of (P_5, Q)-free graphs are either nearly C_5-free or 1-specific graphs.*

Proof. Assume that there is a vertex $v \in V$ such that $G[\overline{N(v)}]$ contains C_5 C, say with vertex set $\{v_1, \dots, v_5\}$ and edges $v_i v_{i+1}$, $i \in \{1, \dots, 5\}$ (index arithmetic modulo 5).

Since G is (P_5, Q)-free, a straightforward case analysis shows that in $N(v)$, C has no 1- and no 2-vertex, and if x is a 3-vertex of C then x is adjacent to v_i, v_{i+1} and v_{i+3}.

Let U denote the set of 5-vertices of C, let F denote the set of 4-vertices of C in $N(v)$ and let $N_{i,i+1,i+3}$ denote the set of 3-vertices of C in $N(v)$ being adjacent to v_i, v_{i+1} and v_{i+3}. Since G is (P_5, Q)-free, each of these sets is a clique and $U ① F$, $U ① N_{i,i+1,i+3}$, and $F ① N_{i,i+1,i+3}$. Let $D = \bigcup_{i=1}^{5} N_{i,i+1,i+3}$ be the set of all 3-vertices of C in $N(v)$. We claim that $U \cup F \cup D$ is a cutset between v and C. Assume that this is not the case. Then there is a (shortest) path P in $G[V \setminus (U \cup F \cup D)]$ between v and C via a 0-vertex $x \in N(v)$. Since G is P_5-free, the neighbor $y \notin N(v)$ of x on P must be a neighbor of C but since G is P_5-free, y cannot distinguish C, i.e., $y \in U$ — a contradiction. Thus, $U \cup F \cup D$ is a cutset but G was assumed to have no clique cutset. Thus, D is no clique, i.e., there are 3-vertices x, y in $N(v)$ with $xy \notin E$. The only possibility is $x \in N_{i,i+1,i+3}$, $x \in N_{i+2,i+3,i+5}$ for some i but then G contains a co-xC_6 with vertices $v, x, y, v_1, \dots, v_5$ which shows Lemma 3. □

Proof of Theorem 5. Assume that G is a (P_5, Q)-free graph without clique cutset. Then by Lemma 1, it is nearly house-free. If it contains a co-xC_6 then, by Lemma 2, it is 1-specific. Now assume that G contains no co-xC_6. Then, by the proof of Lemma 3, it is nearly C_5-free which means that it is nearly $(P_5, \overline{P_5}, C_5)$-free which shows Theorem 5. □

In [13], it has been observed that $(P_5, \overline{P_5}, C_5)$-free graphs are perfectly orderable, and a perfect order of such a graph can be constructed in linear time by a degree order of the vertices. Thus, also for $\overline{G}$, a perfect order can be obtained in linear time. In [23], Hoàng gave an $\mathcal{O}(nm)$ time algorithm for the Maximum Weight Clique problem on a perfectly ordered graph. This means that the MWS problem on $(P_5, \overline{P_5}, C_5)$-free graphs can be solved in time $\mathcal{O}(nm)$ and consequently, it can be solved on nearly $(P_5, \overline{P_5}, C_5)$-free graphs in time $\mathcal{O}(n^2 m)$.

Now, by Theorem 5, MWS is solvable in time $\mathcal{O}(n^2 m)$ time on atoms of (P_5, Q)-free graphs. Then the clique separator approach of Tarjan implies:

Corollary 4. *The MWS problem can be solved in time $\mathcal{O}(n^4 m)$ on graphs whose atoms are (P_5, Q)-free.*

Note that this class is not restricted to (P_5, Q)-free graphs; it is only required that the atoms are (P_5, Q)-free. Thus, it contains, for example, all chordal graphs. The same remark holds for the other sections.

$(P_5, \overline{P_5}, C_5)$-free graphs are also those graphs which are Meyniel and co-Meyniel (see [10]); Meyniel graphs can be recognized in time $\mathcal{O}(m^2)$ [31]. Thus, nearly $(P_5, \overline{P_5}, C_5)$-free graphs can be recognized in time $\mathcal{O}(n^5)$ (which is even better than $\mathcal{O}(n^4 m)$).

Since $(P_5, \overline{P_5}, C_5)$-free graphs are weakly chordal, another consequence of Theorem 5 is:

Corollary 5. *Atoms of (P_5, Q)-free graphs are either nearly weakly chordal or 1-specific.*

Note that weakly chordal graphs can be recognized in time $\mathcal{O}(m^2)$ [6,22]. Thus, recognizing whether G is nearly weakly chordal can be done in time $\mathcal{O}(nm^2)$.

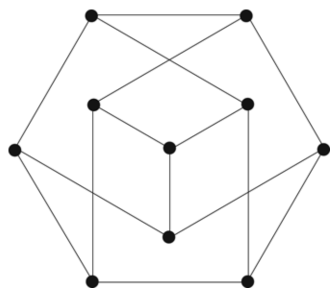

Fig. 4. The Petersen graph B.

The time bound for MWS on weakly chordal graphs, however, is $\mathcal{O}(n^4)$ [32], and thus, worse than the one for $(P_5, \overline{P_5}, C_5)$-free graphs.

An extension of Theorem 4 for various larger subclasses of P_5-free graphs is given in [9].

6. (P_6, C_4)-free graphs and clique separators

By using augmenting techniques, Mosca [28] has shown that the MS problem can be solved in time $\mathcal{O}(n^4)$ on (P_6, C_4)-free graphs. The complexity of the weighted MWS problem remained open in [28]. Here we show that the concept of clique separators can also be applied to (P_6, C_4)-free graphs. As a first step, we show:

Theorem 6. *(P_6, C_4)-free graphs without clique cutset are nearly perfect.*

Proof. Let G be a (P_6, C_4)-free graph without clique cutset. Then G contains no C_k for $k \geq 7$ and no $\overline{C_k}$ for $k \geq 6$.

Now we use again the Strong Perfect Graph Theorem. In order to show that (P_6, C_4)-free graphs without clique cutset are nearly perfect, it is sufficient to show that these graphs are nearly C_5-free.

Assume to the contrary that there is a vertex v in G whose antineighborhood $\overline{N}(v)$ contains a C_5 C, say with vertices $v_1, \ldots, v_5$ and edges $v_i v_{i+1}$, $i \in \{1, \ldots, 5\}$ (index arithmetic modulo 5).

Since G is (P_6, C_4)-free, no vertex $x \in N(v)$ is a k-vertex, $k \in \{1, 2, 4\}$, for C, and 3-vertices have consecutive neighbors in C. Let M_k denote the set of k-vertices for C, and let $M_k(v)$ denote the set of k-vertices for C in $N(v)$, $k \in \{0, 3, 5\}$.

Since G is C_4-free, M_5 is a clique and has a join to the set M_3 of 3-vertices for C. Moreover, $M_3(v)$ is a clique since for any two vertices $x, x' \in M_3(v)$, there is a vertex $y \in C$ that is adjacent to both x and x' since x, x' have three neighbors in C. Now, $xx' \in E$ for otherwise v, x, x', y induce a C_4.

Let $R := M_5 \cup M_3(v)$ which is a clique as shown above. Since G has no clique cutset, there is a path between v and C avoiding R, i.e., there is a 0-vertex $x \in N(v)$ having a neighbor $y \in \overline{N}(v)$ but every such neighbor y must be a 5- or 0-vertex for C since G is P_6-free, and if y is a 5-vertex then $y \in R$. The same argument holds for 0-vertex y and a neighbor $y' \in \overline{N}(v)$ of y which is closer to C. Thus, the vertex v together with the 0-vertex neighbors of v on one hand and C on the other hand are in different components of $G[V \setminus R]$ — contradiction. $\square$

Since MWS is solvable in polynomial time for perfect graphs [19], Tarjan's clique separator approach implies:

Corollary 6. *The MWS problem for (P_6, C_4)-free graphs is solvable in polynomial time.*

A more detailed structure analysis of (P_6, C_4)-free graphs is given by the following Theorem 7. For this purpose, we define 2-specific graphs as follows. Let a *clique-C_6* denote the result of substituting cliques into the vertices of a C_6. Fig. 4 shows the Petersen graph B which is an extension of the C_6 by four other vertices. Let B' denote the graph resulting from B by adding a universal vertex to B, i.e., a vertex which is adjacent to all vertices of B. A graph is 2-*specific* if it results from substituting cliques of arbitrary size into vertices of the graph B' or is an induced subgraph of this one (obtained by substituting some cliques of size 0).

Theorem 7. *(P_6, C_4)-free graphs without clique cutset are either nearly chordal or 2-specific.*

Proof. Let G be a (P_6, C_4)-free graph without clique cutset. By Theorem 6, G is nearly C_5-free, i.e., every C_5 in G is dominating.

Now, assume that G is not nearly chordal. Since G is P_6-free, G contains no induced cycle C_k, $k \geq 7$. Thus, there is a vertex v such that $\overline{N}(v)$ contains a C_6 C, say with vertices $v_1, \ldots, v_6$ and edges $v_i v_{i+1}$, $i \in \{1, \ldots, 6\}$ (index arithmetic modulo 6). By Theorem 6, $G[\overline{N}(v)]$ is C_5-free.

A simple case analysis shows that C has no 1- and no 5-vertex. Moreover, in $N(v)$, C has no 3-vertex, and in $\overline{N}(v)$, C has no 2- and no 4-vertex, 2-vertices in $N(v)$ have diametral neighbors v_i, v_{i+3}, $i \in \{1, 2, 3\}$, in C, and 3- and 4-vertices have consecutive neighbors in C.

Let $T_{i,i+3}$ denote the corresponding sets of 2-vertices being adjacent to v_i and v_{i+3}, $i \in \{1, 2, 3\}$, let H_i denote the corresponding sets of 3-vertices being adjacent to v_{i-1}, v_i and v_{i+1}, $i \in \{1, \ldots, 6\}$, and let $F_{i,i+3}$ denote the corresponding sets of 4-vertices being adjacent to v_i, v_{i+1}, v_{i+2} and v_{i+3}, $i \in \{1, \ldots, 6\}$. Let U denote the set of all 6-vertices of C.

Note that all these sets are cliques since G is C_4-free. Moreover, U has a join to any other of these sets, there is a pairwise co-join between the 2-vertex sets since G is C_4-free, there is a join between consecutive 3-vertex sets since G is P_6-free, there is a co-join between nonconsecutive sets of 3-vertices since G is C_4-free and $G[\overline{N}(v)]$ is C_5-free, there is a pairwise join between sets of 4-vertices, and there is a pairwise join between any set of 2-vertices and any set of 4-vertices since G is C_4-free. Moreover, 0-vertices are not adjacent to 3-vertices since G is P_6-free.

Let R be the set of all 2-, 4- and 6-vertices of C. Since G contains no clique cutset but R separates v from C as R does in the proof of Theorem 6, R is not a clique. This means that at least two of the 2-vertex sets are nonempty. Without loss of generality, let $x \in T_{1,4}$ and $y \in T_{2,5}$ be such 2-vertices. Note that x (y, respectively) forms a (dominating) C_5 with some vertices in C. Hence, every 0-vertex in $\overline{N}(v)$ is adjacent to x and y but then G contains a C_4; thus, there are no 0-vertices in $\overline{N}(v)$. Moreover, there are no 6-vertices in $\overline{N}(v)$ since G is C_4-free, and there are no 4-vertices: If $z \in F_{i,i+3}$ then $xz \in E$ and $yz \in E$ since x and z (y and z, respectively), have a common neighbor in C. Now, if $z \in F_{1,4}$ then zv_4v_5y is a C_4 — contradiction. Thus, $F_{1,4} = \emptyset$ and similarly for the other 4-vertex sets.

Note that the 3-vertices together with C form a clique-C_6.

Let O_v denote the set of 0-vertices in $N(v)$. Since every C_5 is dominating, there is a join between O_v and the 2-vertex sets. Now, since G is C_4-free, $\{v\} \cup U \cup O_v$ must be a clique since $xy \notin E$.

Thus, $V(G)$ has a partition into a clique $\{v\} \cup U \cup O_v$, a clique-C_6 and (at most) three cliques of 2-vertices such that there is a join between U and all other vertices, there is a join between O_v and the sets of 2-vertices, and there are joins between $F_{i,i+3}$ and H_i, H_{i+3} of the clique-C_6. All other connections are co-join. Now, G is a 2-specific graph. □

The time bounds given in [33] imply:

Corollary 7. *The MWS problem for* (P_6, C_4)*-free graphs is solvable in time* $\mathcal{O}(n^3 m)$.

Note that, in general, 2-specific graphs are not prime.

Corollary 8. *Prime* (P_6, C_4)*-free graphs with at least* 11 *vertices and without clique cutset are nearly chordal.*

7. Combining decomposition by clique cutsets and by homogeneous sets

In this section, we combine the two approaches, namely decomposition by clique cutsets and decomposition by homogeneous sets, to a binary decomposition tree which gives a refinement of the decompositions obtained separately.

Consider a graph $G = (V, E)$. Let $G[V_i]$ be the subgraph of G induced by $V_i \subseteq V$. We study the decomposition ϕ of a graph G defined as follows. If G has a clique cutset $C \subset V$, then G is decomposed into subgraphs $G_1 = G[V_1]$ and $G_2 = G[V_2]$ where $V = V_1 \cup V_2$ and $C = V_1 \cap V_2$. Whitesides [35] (see also Tarjan [33]) showed that given a decomposition of a graph G into two graphs G_1, G_2 as above, if the MWS problem for G_1, G_2 can be computed in polynomial time, then so can the problem for G.

If G has a homogeneous set H, then G is decomposed into induced subgraphs $G_1 = G[V_1]$ and $G_2 = G[V_2]$ where $V_1 = H$ and $V_2 = V - H \cup \{h\}$ for some vertex h in H. It is easy to see that given a decomposition of a graph G into two graphs G_1, G_2 as above, if the MWS problem for G_1, G_2 can be computed in polynomial time, then so can the problem for G.

We can recursively decompose G_1 and G_2 in the same way, until we obtain ϕ-prime graphs (i.e., graphs that have no clique cutset, and no homogeneous set). This decomposition can be represented by a binary tree $T(G)$ whose root

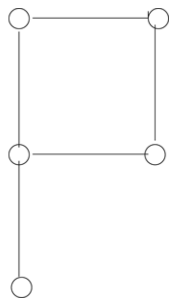

Fig. 5. The graph P.

is G, the two children of G are G_1 and G_2, which are in turn the roots of subtrees representing the decompositions of G_1 and G_2. Each leaf of $T(G)$ corresponds to an induced ϕ-prime subgraph of G.

Theorem 8. *For any graph G, $T(G)$ contains $\mathcal{O}(n^2)$ nodes.*

Proof. We will show that each internal node of $T(G)$ can be labeled with a distinct 2-tuple (a, b) where a, b are two vertices of G. We only need to label internal nodes that correspond to graphs with at least three vertices.

Let G_X denote the induced subgraph of G that corresponds to an internal node X of $T(G)$. If G_X is decomposed by a clique cutset C into two graphs G_1, G_2, then label X with (a, b) where a is any vertex $G_1 - C$, and b is any vertex $G_2 - C$ (we say X is a *node of type* 1). If G_X has a homogeneous set H, and G is decomposed into subgraphs $G_1 = G[V_1]$ and $G_2 = G[V_2]$ where $V_1 = H$ and $V_2 = V - H \cup \{h\}$ for some vertex h in H, then we label X with (a, b), where a is any vertex in $V - H$, and b is a vertex in $H - \{h\}$ (we say X is a *node of type* 2). Now we show that each internal node in $T(G)$ has a distinct 2-tuple.

Assume there are two nodes A, B in $T(G)$ with the same 2-tuple (x, y), in particular, we have $x, y \in G_A \cap G_B$. Suppose first that B is a descendant of A. Our choice of the labels implies that, whether A is of type 1 or 2, there is at least one vertex in the label of G_A that does not belong to G_B, a contradiction.

Now, we may assume A is not a descendant of B and B is not a descendant of A. Let X be the lowest common ancestor of A and B in $T(G)$. For simplicity, we may assume that A (B) either is the left (right) child of X, or is a descendant of the left (right) child of X. If X is a node of type 2, then A and B can have at most one vertex in common, and thus cannot have the same 2-tuple. So, X must be a node of type 1 with a clique cutset C. We thus have $x, y \in C$ implying xy is an edge. This is a contradiction since we chose x to be nonadjacent to y. □

Corollary 9. *If the MWS problem can be solved in polynomial time for every ϕ-prime subgraph of a graph G, then so can the problem for G.* □

As an example, we will study the corresponding decomposition of (P_5, P)-free graphs.

8. Prime (P_5, P)-free graphs are nearly perfect

Denote by P the graph shown in Fig. 5.

In [11,25], a (robust, see [11]) polynomial time solution for the (unweighted) Maximum Stable Set Problem on (P_5, P)-free graphs was given. The complexity of the MWS problem on (P_5, P)-free graphs remained an open question in [11,25]. In this section, we will show that prime (P_5, P)-free graphs are nearly perfect which implies that the MWS problem is solvable in polynomial time on this graph class.

Theorem 9. *Prime (P_5, P)-free graphs are nearly hole-free.*

Proof. Let G be a prime (P_5, P)-free graph. Since G is P_5-free, it is also C_k-free for $k \geq 6$. Now assume that for some vertex v, $G[V \setminus N[v]]$ contains a C_5 C with vertices $\{v_1, \ldots, v_5\}$ and edges $v_i v_{i+1}$, $i \in \{1, \ldots, 5\}$ (index arithmetic modulo 5). Then C has no 1-vertex since G is P_5-free, and C has no 2-vertex since G is P_5- and P-free. Moreover, 0-vertices of C are nonadjacent to 3- and 4-vertices of C.

Let R denote the set of 0-vertices of C, let D denote the set of all 3- and 4-vertices of C, let U denote the set of all 5-vertices of C, let U_1 denote the U-vertices with a neighbor in R and let $U_0 = U \setminus U_1$. Since G is P-free, we have:

Hence, $D \cup U_0 \cup \{v_1, \ldots, v_5\}$ is a module in G. Since G is prime, this module must be trivial, i.e., $D \cup U_0 \cup \{v_1, \ldots, v_5\} = V$ which in particular means that C has no 0-vertices but v is a 0-vertex for C — contradiction. □

In a similar but simpler way one shows:

Proposition 1. *If in a prime (P_5, P)-free graph, for a vertex x, $G[\overline{N}(x)]$ contains a C_4 C then every vertex in $N(x)$ is either a 4-vertex or a 0-vertex for C.*

Next, we are going to show that prime (P_5, P)-free graphs are nearly odd-antihole-free. For this purpose, we switch to the complement graph.

Claim 8.1. *Let G be a (co-P_5, co-P)-free graph without odd cycle $C_5, \ldots, C_{2k-1}$, $k \geq 3$. Then any C_{2k+1}, $k \geq 2$, in G has no ℓ-vertex for $\ell \in \{3, 4, \ldots, 2k\}$.*

Proof. Let C be a C_{2k+1} in G with vertices $v_1, \ldots, v_{2k+1}$ and edges $v_i v_{i+1}$, $i \in \{1, \ldots, 2k+1\}$ (index arithmetic modulo $2k+1$). If a vertex x not in C has consecutive neighbors v_i, v_{i+1} in C then x is adjacent to all vertices v_i, $i \in \{1, \ldots, 2k+1\}$, since G is (co-P_5, co-P)-free. The other case in which a vertex x not in C has at least three neighbors in C, and that no two of them are consecutive is also impossible since G is $C_{2k'+1}$-free for $k' < k$. □

Claim 8.2. *Let G be a (co-P_5, co-P)-free graph and let C be a C_{2k+1}, $k \geq 2$, in G having no ℓ-vertex for $\ell \in \{3, 4, \ldots, 2k\}$. Let U denote the set of $(2k+1)$-vertices of C. Then every $(2k+1)$-vertex for C dominates the connected component of $G[V \setminus U]$ containing C.*

Proof. Since G is co-P-free, every $(2k+1)$-vertex x for C is adjacent to every 1- and every 2-vertex for C. Moreover, if $yy' \in E$ is an edge of vertices not in C without edges between y, y' and v_i, v_{i+1} for some $i \in \{1, 2, \ldots, 2k+1\}$ and $xy \in E$ then also $xy' \in E$ which shows Claim 8.2. □

Claim 8.3. *Let G be a (co-P_5, co-P)-free graph and let C be a C_{2k+1}, $k \geq 2$, in $N(v)$ for some vertex v. Then every vertex in $\overline{N}(v)$ is either a $(2k+1)$-vertex or a 0-vertex for C.*

Proof. Let $x \in \overline{N}(v)$. As in the proof of Claim 8.1, if x has consecutive neighbors in C then x is a $(2k+1)$-vertex for C. Now assume that x has no consecutive neighbors in C. If x has one or two neighbors in C then obviously, there is a co-P or co-P_5 in G with v and some vertices in C. Now assume that x has at least three (namely, nonconsecutive) neighbors in C. Then, since C has odd length, there are consecutive v_i, v_{i+1} with $xv_i \notin E$, $xv_{i+1} \notin E$. Moreover, there is v_j with $xv_j \in E$ and $j \notin \{i-1, i+2\}$ but now, x, v_j, v, v_i, v_{i+1} is a co-P — contradiction. Thus, x is either a $(2k+1)$-vertex or a 0-vertex for C. □

Theorem 10. *Prime (P_5, P)-free graphs are nearly odd-antihole-free.*

Proof. Assume not; let G be a prime (P_5, P)-free graph, and let v be a vertex such that $G[\overline{N}(v)]$ contains an odd antihole $\overline{C_{2k+1}}$ C with vertices $v_1, \ldots, v_{2k+1}$ and co-edges $v_i v_{i+1}$, $i \in \{1, \ldots, 2k+1\}$ (index arithmetic modulo $2k+1$).

Then in the prime (co-P_5, co-P)-free graph $G' = \overline{G}$, v is a vertex such that $G'[N(v)]$ contains an odd hole C_{2k+1} C with vertices $v_1, \ldots, v_{2k+1}$ and edges $v_i v_{i+1}$, $i \in \{1, \ldots, 2k+1\}$. By Theorem 9, $k \geq 3$. Let k be the smallest value such that for a vertex v, $G'[N(v)]$ contains an odd hole C_{2k+1} C. Let U_v denote the set of $(2k+1)$-vertices for C in $N(v)$.

Now consider the connected component R in $G'[N(v) \setminus U_v]$ containing C. By Claim 8.1, C has only 0-, 1-, 2- and $(2k+1)$-vertices in $N(v)$. By Claim 8.3, every vertex $x \notin N[v]$ is either a $(2k+1)$-vertex or a 0-vertex for C. By Claim 8.2, every $(2k+1)$-vertex $x \in N[v]$ for C and every $(2k+1)$-vertex $x \notin N[v]$ dominates R. By the definition of R, no 0-vertex $x \in N(v) \setminus R$ has an edge to any vertex in R, and since G' is co-P-free, no 0-vertex $x \notin N[v]$ has an edge to any vertex in R. This implies that R is a homogeneous set — contradiction. □

Corollary 10. *Prime (P_5, P)-free graphs are nearly perfect.*

Proof. By Theorem 9, prime (P_5, P)-free graphs are nearly hole-free, and by Theorem 10, these graphs are nearly odd-antihole-free. Thus, by the Strong Perfect Graph Theorem by Chudnovsky, Robertson, Seymour and Thomas [12], these graphs are nearly perfect. □

Corollary 11. *The Maximum Weight Stable Set Problem can be solved in polynomial time for (P_5, P)-free graphs.*

9. (P_5, P)-free graphs and clique separators

In this section, we prove the following

Theorem 11. *Prime (P_5, P)-free graphs without clique cutset are nearly chordal.*

Proof. Let G be a prime (P_5, P)-free graph without clique cutset. Assume that G is not nearly chordal, i.e., there is a vertex v such that $G[\overline{N}(v)]$ is not chordal. Then, by Theorem 9, $G[\overline{N}(v)]$ contains a C_4 C, say, with vertices v_1, v_2, v_3, v_4 and edges $v_i v_{i+1}$, $i \in \{1, 2, 3, 4\}$. By Proposition 1, every vertex in $N(v)$ is a 0- or 4-vertex for C. Let N_1 denote the set of 0-vertices for C in $N(v)$, and let U denote the set of 4-vertices for C and $U_1 = U \cap N(v)$, $U_2 = U \cap \overline{N}(v)$.

Claim 9.1. *U separates v and C.*

Proof. Suppose there is a (shortest) path P between v and C in $G[V \setminus U]$. Then, since G is P_5-free, P has exactly four vertices v, x, y, v_i, for some $i \in \{1, 2, 3, 4\}$, with $x \in N_1$ and $y \in \overline{N}(v)$. Note that y cannot be a 1-, 2-, or 3-vertex for C since G is (P_5, P)-free. Thus, y is a 4-vertex for C, i.e., $y \in U_2$ — contradiction. This shows the claim.

Claim 9.2. $U_1 ① U_2$.

Proof. Suppose there are $x \in U_1$ and $y \in U_2$ with $xy \notin E$. Then vxv_1v_3y is a P — contradiction. This shows the claim.

Claim 9.3. *If $x \in U_2$ has a nonneighbor $y \in U_2$ then x has no neighbors in the set N_1 of 0-vertices in $N(v)$.*

Proof. Suppose there are $x, y \in U_2$ with $xy \notin E$ and there is $w \in N_1$ such that $xw \in E$. Then, if $wy \in E$, $vwxyv_1$ is a P, and if $wy \notin E$, wxv_1v_3y is a P — contradiction. This shows the claim.

Now let $R \subseteq U$ be a minimal separator of v and C. Obviously, $U_1 \subseteq R$ holds.

Claim 9.4. *Every vertex $u \in R \cap U_2$ is adjacent to a vertex in N_1.*

Proof. Suppose u is not adjacent to any vertex in N_1. Let C_v be the component containing v of the graph $G[V \setminus R]$. Since R is a minimal separator, u has a neighbor in C_v. Consider a shortest path R from v to u whose interior vertices lie entirely in C_v. Since G contains no P_5 and u is not adjacent to any vertex in N_1, R is a P_4. Now, R and v_1 (in C) form a P_5. This shows the claim.

Claims 9.3 and 9.4 imply $R \cap U_2$ is a clique.

By Claim 9.2, and since G has no clique cutset, $R \cap U_1$ must contain a pair x, y of nonadjacent vertices $xy \notin E$. Let A be the connected component in $\overline{G}[U_1]$ containing x and y. Since G is prime, A is not a homogeneous set, i.e., there is a vertex $z \notin A$ distinguishing a pair x' and y' of nonadjacent vertices in A, say, $x'z \in E$ and $y'z \notin E$. Actually, $z \notin U_1$. If z is a 0-vertex in $N(v)$ then $zx'y'v_1v_3$ is a P, and if $z \in \overline{N}(v)$ then, by Claim 9.2, z is not a 4-vertex for C. Let c be a nonneighbor of z in C. Then $vx'cy'z$ is a P — a contradiction which shows Theorem 11. □

This leads to Corollary 11 in yet another way with better time bound:

Corollary 12. *The MWS problem for (P_5, P)-free graphs is solvable in time $\mathcal{O}(n^3 m)$.*

Corollary 12 follows from Corollary 9.

10. Conclusion

In this paper, we give new applications of the clique separator approach, combine it with the decomposition by homogeneous sets and improve some known polynomial time results for the unweighted Maximum Stable Set problem which were obtained by various methods such as struction, augmenting and elimination of α-redundant vertices to the weighted MWS problem.

It remains a challenging task to study under which conditions a method works for the unweighted MS problem can be replaced by the decomposition which is studied in this paper. Moreover, the combined decomposition should be applied to other cases where augmenting and other methods for MS work well. In particular, it might be interesting whether the $\mathcal{O}(n^7)$ time augmenting algorithm for the MS problem on (P_7, P)-free graphs given by Alekseev and Lozin in [3] can be explained in terms of our approach.

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